6.3 – Gram-Schmidt Process

Definition: A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be **orthonormal**.

Just with the Euclidean inner product, we can normalize a vector to obtain a unit vector: $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

$\langle \vec{v}_i, \vec{v}_j \rangle = 0,\ i \neq j$

Theorem 6.3.1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

pf: Suppose S is as described and $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n = \vec{0}$ and consider $\langle k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n, \vec{v}_i \rangle$, $1 \le i \le n$.

$\Rightarrow \langle \vec{0}, \vec{v}_i \rangle = 0$.

But $\langle k_1\vec{v}_1, \vec{v}_i \rangle + \langle k_2\vec{v}_2, \vec{v}_i \rangle + \cdots + \langle k_i\vec{v}_i, \vec{v}_i \rangle + \cdots + \langle k_n\vec{v}_n, \vec{v}_i \rangle$

$= \langle k_i\vec{v}_i, \vec{v}_i \rangle$, since $\langle \vec{v}_k, \vec{v}_i \rangle = 0$ if $k \neq i$.

So $\langle k_i\vec{v}_i, \vec{v}_i \rangle = 0 \Rightarrow k_i \langle \vec{v}_i, \vec{v}_i \rangle = 0$.

But $\vec{v}_i \neq \vec{0} \Rightarrow k_i = 0\ \forall i$. So S is linearly independent.

Definition: In an inner product space, a basis comprising orthogonal vectors is an **orthogonal basis**, and a basis comprising orthonormal vectors is an **orthonormal basis**.

$R^3: \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ - orthonormal basis

orthogonal basis $\{(3,4,0), (4,-3,0), (0,0,1)\}$

orthonormal basis $\{(\frac{3}{5}, \frac{4}{5}, 0), (\frac{4}{5}, -\frac{3}{5}, 0), (0,0,1)\}$

#6 Show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space.

$$\begin{array}{cc} & \begin{array}{ccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{array} \\ A = & \begin{bmatrix} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix} \end{array}$$

$\vec{c}_1 \cdot \vec{c}_2 = -\frac{1}{10} + \frac{1}{10} + 0 = 0$, $\vec{c}_1 \cdot \vec{c}_3 = \frac{1}{15} + \frac{1}{15} - \frac{2}{15} = 0$

$\vec{c}_2 \cdot \vec{c}_3 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$

So the 3 orthogonal columns are a basis for $\mathbb{R}^3$.

Orthonormal basis $\Rightarrow \|\vec{v}_i\| = 1$

$\cancel{\|\vec{c}_1\| = \sqrt{\frac{1}{25} + \frac{1}{25} + \frac{1}{25}}}$ use $5\vec{c}_1 = \langle 1, 1, 1 \rangle$

* Scale the vectors first *

$\vec{v}_1 = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$ $\vec{c}_2 = \langle -\frac{1}{2}, \frac{1}{2}, 0 \rangle \rightarrow \langle 1, -1, 0 \rangle$

$\vec{v}_2 = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \rangle$

$\vec{v}_3 = \langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \rangle$ orthonormal basis: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Theorem 6.3.2

a) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

these are projections

b) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then $\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$.

$(\vec{u})_S = (\langle \vec{u}, \vec{v}_1 \rangle, \langle \vec{u}, \vec{v}_2 \rangle, \ldots, \langle \vec{u}, \vec{v}_n \rangle) \in \mathbb{R}^n$

Pf: Let $\vec{u} \in V$. Then since S is a basis for V,

$\exists\, c_i \ni \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \Rightarrow \langle \vec{u}, \vec{v}_i \rangle$, $0 \le i \le n$

is $\langle c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \vec{v}_i \rangle = c_i \langle \vec{v}_i, \vec{v}_i \rangle$

because S is orthogonal. Hence $\langle \vec{u}, \vec{v}_i \rangle = c_i \langle \vec{v}_i, \vec{v}_i \rangle$

$$\Rightarrow c_i = \frac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}.$$

If S is an orthonormal basis, then $\|\vec{v}_i\|^2 = 1$,

so $c_i = \langle \vec{u}, \vec{v}_i \rangle$.

#8 Use Theorem 6.3.2(b) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $\mathbf{v}_3 = (0, 0, 1)$.

Note: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{u}, \vec{v}_3 \rangle \vec{v}_3$

$\vec{u} = -\frac{37}{5}\vec{v}_1 - \frac{9}{5}\vec{v}_2 + 4\vec{v}_3$

#12 Find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in Exercise 8.

$(\vec{u})_S = \left(-\frac{37}{5}, -\frac{9}{5}, 4\right)$

Theorem 6.3.3 Projection Theorem (generalization of Theorem 3.3.2)
If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$.

Definition: If W is a finite-dimensional subspace of an inner product space V and a vector $\mathbf{u}$ in V is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$, then $\mathbf{w}_1$ is called the **orthogonal projection of u on W** and is denoted by $\mathbf{w}_1 = \text{proj}_W \mathbf{u}$ and $\mathbf{w}_2$ is called the **orthogonal projection of u on $\mathbf{W}^\perp$** and is denoted by $\mathbf{w}_2 = \text{proj}_{W^\perp} \mathbf{u}$. The vector $\mathbf{w}_2$ is also called the **component of u orthogonal to W**.

Theorem 6.3.4 Let W be a finite-dimensional subspace of an inner product space V. (see diagram at end of notes)
a) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthogonal basis for W and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \dots \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

b) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthonormal basis for W and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \dots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

Pf: From 6.3.3, we have $\vec{u} = \vec{w}_1 + \vec{w}_2$.

$$\text{proj}_W \vec{u} = \vec{w}_1 = \frac{\langle \vec{w}_1, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{w}_1, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \dots + \frac{\langle \vec{w}_1, \vec{v}_k \rangle}{\|\vec{v}_k\|}\vec{v}_2$$

Since $\vec{w}_2 \in W^\perp$, $\langle \vec{w}_2, \vec{v}_i \rangle = 0$, $1 \le i \le k$.

So $$\text{proj}_W \vec{u} = \vec{w}_1 = \frac{\langle \vec{w}_1 + \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \dots + \frac{\langle \vec{w}_1 + \vec{w}_2, \vec{v}_k \rangle}{\|\vec{v}_k\|^2}\vec{v}_k$$

$$= \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \dots + \frac{\langle \vec{u}, \vec{v}_k \rangle}{\|v_k\|^2}\vec{v}_k$$

If the basis is orthonormal, then $\|\vec{v}_i\|^2 = 1$, $i \le 1 \le k$.

#24 The vectors $\mathbf{v}_1 = (0,\ 1,\ -4,\ -1)$ and $\mathbf{v}_2 = (3,\ 5,\ 1,\ 1)$ are orthogonal with respect to the Euclidean inner product on R^4. Find the orthogonal projection $\mathbf{w}_1$ of $\mathbf{b} = (1,\ 2,\ 0,\ -2)$ on the subspace W spanned by these vectors. [Extension of the exercise] Then find $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

$$\vec{w}_1 = \operatorname{proj}_W \vec{b} = \frac{\langle \vec{b}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{b}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\vec{w}_1 = \frac{4}{18}(0,1,-4,-1) + \frac{11}{36}(3,5,1,1)$$

$$\vec{w}_1 = \left(\frac{11}{12}, \frac{7}{4}, -\frac{7}{12}, \frac{1}{12}\right)$$

$$\vec{b} = \vec{w}_1 + \vec{w}_2 \Rightarrow \vec{w}_2 = \vec{b} - \vec{w}_1 = \left(\frac{1}{12}, \frac{1}{4}, \frac{7}{12}, -\frac{25}{12}\right)$$

Note: $\vec{w}_2 \cdot \vec{v}_1 = \frac{1}{4} - \frac{7}{3} + \frac{25}{12} = 0$, $\vec{w}_2 \cdot \vec{v}_2 = 0$

#25 The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1,\ 2,\ 0,\ -1)$ onto the subspace W spanned by these vectors.

$\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$, $\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$

$$\operatorname{proj}_W \vec{b} = \left(\frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}}\right)\vec{v}_1 + \left(\frac{1}{2} + \frac{5}{3} - \frac{1}{6}\right)\vec{v}_2 + \left(\frac{1}{\sqrt{18}} + \frac{4}{\sqrt{18}}\right)\vec{v}_3$$

$$= \left(0, \frac{1}{6}, -\frac{2}{3}, -\frac{1}{6}\right) + \left(1, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{5}{18}, 0, \frac{5}{18}, -\frac{10}{9}\right)$$

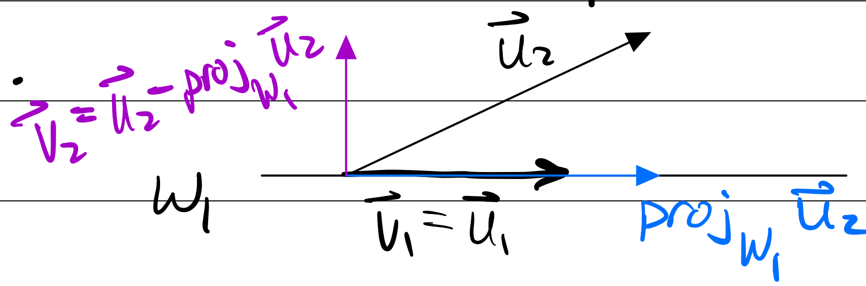
$$= \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)$$

Theorem 6.3.5 (proof outlines the Gram-Schmidt process)
Every nonzero finite-dimensional inner product space has an orthonormal basis.

This is a constructive proof.

~~pf~~: Let W be any nonzero finite-dimensional subspace of an inner product space and suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ is any basis for W. We will produce an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ for W that can be normalized to form an orthonormal basis.

Let $\vec{v}_1 = \vec{u}_1$ and let W_1 be the space spanned by $\vec{v}_1$.

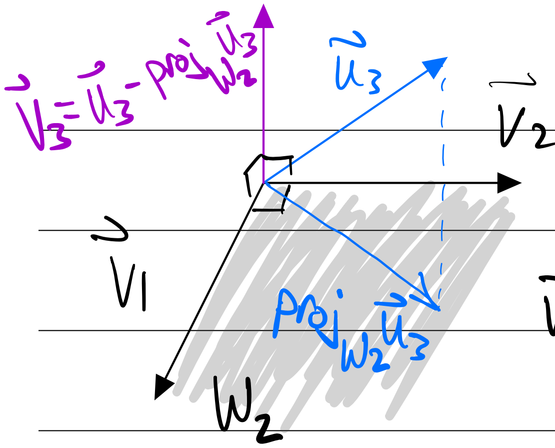


Let $\vec{v}_2 = \vec{u}_2 - \text{proj}_{W_1} \vec{u}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$

We know that $\vec{v}_2 \neq \vec{0}$ because if $\vec{v}_2 = \vec{0}$, then $\vec{u}_2$ is a scalar multiple of $\vec{u}_1$, a contradiction.

By construction, $\vec{v}_2 \perp \vec{v}_1$.

Now let W_2 be the space spanned by $\vec{v}_1$ & $\vec{v}_2$



$$\vec{v}_3 = \vec{u}_3 - \text{proj}_{W_2}\vec{u}_3$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$\vec{v}_3 \perp W_2$ and so is orthogonal to $\vec{v}_1$ & $\vec{v}_2$.

We can likewise find a vector $\vec{v}_4$ that is orthogonal to $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$:

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

and so on until we have r vectors.

Normalizing $\vec{v}_i$ results in an orthonormal basis.

#31 Let R^4 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.
$\mathbf{u}_1 = (0, 2, 1, 0)$, $\mathbf{u}_2 = (1, -1, 0, 0)$, $\mathbf{u}_3 = (1, 2, 0, -1)$, $\mathbf{u}_4 = (1, 0, 0, 1)$

$\vec{v}_1 = \vec{u}_1 = (0, 2, 1, 0)$

$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = (1, -1, 0, 0) - \frac{(-2)}{5}(0, 2, 1, 0)$

$\vec{v}_2 = (1, -\frac{1}{5}, \frac{2}{5}, 0)$

use $\vec{v}_2{}' = (5, -1, 2, 0)$ * Check for orthogonality as you go $\vec{v}_2{}' \perp \vec{v}_1$.

$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2{}'}{\|\vec{v}_2{}'\|^2}\vec{v}_2{}'$

$\vec{v}_3 = (1, 2, 0, -1) - \frac{4}{5}(0, 2, 1, 0) - \cancel{\frac{3}{30}}\,\frac{1}{10}(5, -1, 2, 0)$

$= (1, 2, 0, -1) + (0, -\frac{8}{5}, -\frac{4}{5}, 0) + (-\frac{1}{2}, \frac{1}{10}, -\frac{1}{5}, 0)$

$\vec{v}_3 = (\frac{1}{2}, \frac{1}{2}, -1, -1)$. Use $\vec{v}_3{}' = (1, 1, -2, -2)$

$\vec{v}_3{}' \cdot \vec{v}_1 = 0$ $\quad\quad$ $\vec{v}_3{}' \cdot \vec{v}_2{}' = 0$

$\vec{v}_4 = \vec{u}_4 - \frac{\vec{u}_4 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_4 \cdot \vec{v}_2{}'}{\|\vec{v}_2{}'\|^2}\vec{v}_2{}' - \frac{\vec{u}_4 \cdot \vec{v}_3{}'}{\|\vec{v}_3{}'\|^2}\vec{v}_3{}'$

$\vec{v}_4 = (1, 0, 0, 1) - \frac{0}{5}(0, 2, 1, 0) - \frac{5}{30}(5, -1, 2, 0) - \frac{(-1)}{10}(1, 1, -2, -2)$

$\vec{v}_4 = (1, 0, 0, 1) + (-\frac{5}{6}, \frac{1}{6}, -\frac{1}{3}, 0) + (\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5})$

$\vec{v}_4 = (\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5})$. Use $\vec{v}_4{}' = \cancel{(4, 4, -8, 12)}$

Use $\vec{v}_4{}' = (1, 1, -2, 3)$

$\vec{v}_4{}' \cdot \vec{v}_1 = 0 \quad \vec{v}_4{}' \cdot \vec{v}_2{}' = 0 \quad \vec{v}_4{}' \cdot \vec{v}_3{}' =$

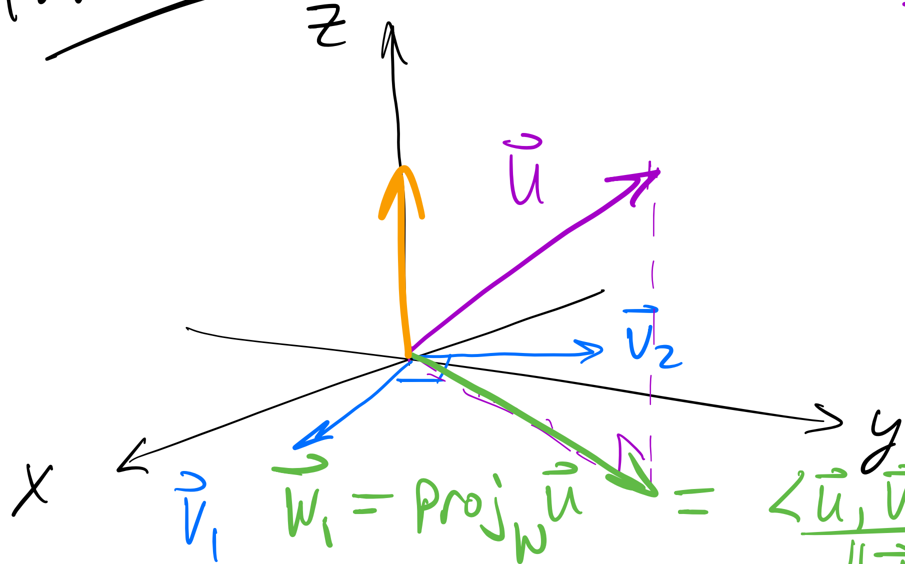
Theorem 6.3.6 (inner product space analog of Theorem 4.6.5 (b))
If W is a finite-dimensional inner product space, then:
a) Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.
b) Every orthonormal set of nonzero vectors in W can be enlarged to an orthonormal basis for W.

Example 6.3.9 Legendre Polynomials
Let the vector space P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$. Apply the Gram-Schmidt process to transform the standard basis $\left\{1, x, x^2\right\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.

<u>Definition:</u> The first three **Legendre polynomials** are 1, x, and $\frac{1}{2}\left(3x^2-1\right)$. They result from transforming the standard basis for P_2 with the inner product $\langle \mathbf{p},\ \mathbf{q}\rangle = \int_{-1}^{1} p(x)q(x)dx$ into an orthogonal basis for P_2 via the Gram-Schmidt process, scaled so they have a value of 1 at $x=1$.

Diagram for Thm 6.3.4

$\mathbb{R}^3$



$\vec{u} = \vec{w}_1 + \vec{w}_2$

in $\mathbb{R}^2$

W: $\mathbb{R}^2$
(xy-plane)

$$\vec{w}_1 = \text{proj}_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$